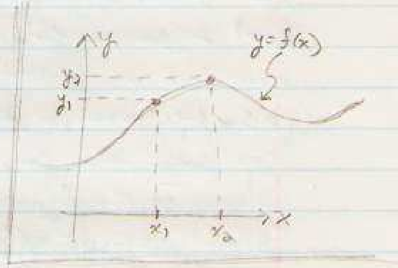
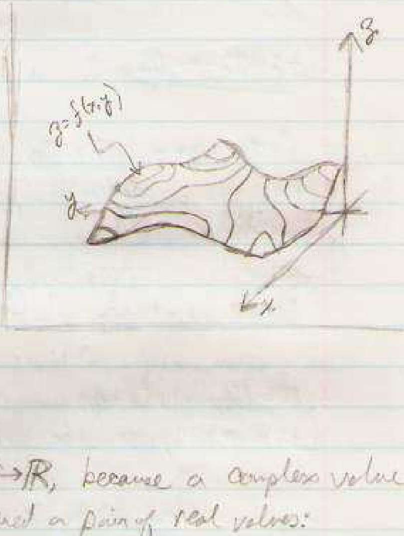
Complex Differentiability (Cauchy-Riemann Equations)

Let's start by wrapping our head around complex-valued functions of complex variables. And to do that, we'll back track, starting all the way back at real valued functions of real variables.



$f: \mathbb{R} \to \mathbb{R}$

For each real value (in the domain of f), the function yields a real value (in the range of f). Taken all together, these values form a curve.



$f: \mathbb{R}^2 \to \mathbb{R}$

This is a function that maps a pair of real values to a single real value. All together, these real values form a surface over the 2-dimensional domain of f.

$f: \mathbb{C} \to \mathbb{R}$

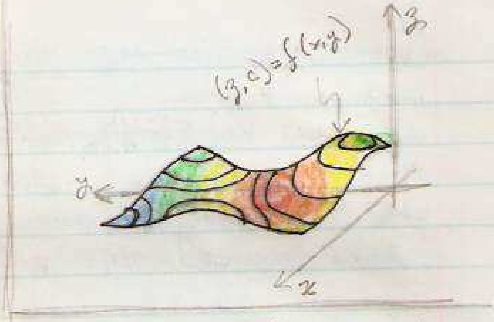
This is basically the same as $\mathbb{R}^2 \to \mathbb{R}$, because a complex value can (for the most part) be considered a pair of real values:

$$z = x + iy \;; \quad z \in \mathbb{C};\ x \in \mathbb{R},\ y \in \mathbb{R}$$

So this function is also a surface, over the complex plane.

$\longrightarrow$

$f: \mathbb{R}^2 \to \mathbb{R}^2$



Now f is a function of two real values, whose value is also a pair of real values. It's a bit hard to continue the geometric visualization at this point: the function produces a surface over a 2-dim domain, but then this surface has some other completely independent property over the plane. In the figure, we use color to represent this second dimension of the function value. Notice how the color is not related to the altitude of the surface: the dimensions are independent.

$f: \mathbb{C} \to \mathbb{C}$

Once again, this is mostly the same as $\mathbb{R}^2 \to \mathbb{R}^2$, except that the algebra is different (as is the nature of complex values).

Differentiation (Cauchy-Riemann Equations)

To differentiate a real valued function of a real variable ($f: \mathbb{R} \to \mathbb{R}$), we take two points on the function & connect them with a line. Then we move them a little closer together & draw a line, & we keep moving them closer & closer. In the limit of this process, the line connecting the two points ends up as a tangent to the curve, & its slope is the derivative of the function at the tangent point.

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We do the same for the derivative of a surface. Take two points on the surface & draw a line between them, & move them closer & closer. We write it using familiar syntax.

$$f'(x,y) = \lim_{(a,b)\to(x,y)} \frac{f(a,b) - f(x,y)}{(a,b)-(x,y)}$$

The thing is, for $f(\cdot)$ to be differentiable at the point (x,y), the limit needs to exist & be the same no matter where you approach from. In other words, no matter where a & b start from in the above limit, the limit must be the same. This is just like we have for differentiation of $\mathbb{R}\to\mathbb{R}$ functions (which is why $f(x)=|x|$ is not differentiable).

So now lets switch back (finally) to $\mathbb{C}\to\mathbb{C}$, & evaluate the derivative at a point.

First, assume that $f(x+iy) = u(x+iy) + iv(x+iy)$ where $u(\cdot)$ & $v(\cdot)$ are each real valued functions of complex variables ($u:\mathbb{C}\to\mathbb{R}$, $v:\mathbb{C}\to\mathbb{R}$).

We'll start by evaluating the limit for differentiation as we approach along a path parallel to the real (x) axis:

$$f'(x+iy) = \lim_{h\to 0} \frac{f(x+h+iy) - f(x+iy)}{h}$$

$$= \lim_{h\to 0} \frac{u(x+h+iy) + iv(x+h+iy) - u(x+iy) - iv(x+iy)}{h}$$

(cont)

$$f'(x+iy) = \lim_{h\to 0} \frac{u(x+h+iy) - u(x+iy)}{h} + i \lim_{h\to 0} \frac{v(x+h+iy) - v(x+iy)}{h}$$

$$\therefore \quad f'(x+iy) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Now, when we evaluate the limit as we approach along a path parallel to the imaginary (y) axis, we get something a little different:

$$f'(x+iy) = \lim_{h\to 0} \frac{f(x+i(y+h)) - f(x+iy)}{ih}$$

$$= \lim_{h\to 0} \frac{u(x+i(y+h)) + iv(x+i(y+h)) - u(x+iy) - iv(x+iy)}{ih}$$

$$= \lim_{h\to 0} \frac{u(x+i(y+h)) - u(x+iy)}{ih} + i \lim_{h\to 0} \frac{v(x+i(y+h)) - v(x+iy)}{ih}$$

$$= \frac{1}{i} \lim_{h\to 0} \frac{u(x+i(y+h)) - u(x+iy)}{h} + \frac{i}{i} \lim_{h\to 0} \frac{v(x+i(y+h)) - v(x+iy)}{h}$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\therefore \quad f'(x+iy) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \qquad \left(\text{because } \frac{1}{i} = -i\right)$$

So we have two expressions for $f'(x,y)$, which must be equal:

$$f'(x+iy) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Setting the real & imaginary parts equal to eachother we get the <u>Cauchy-Riemann</u> equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad + \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

And the point of all this is that for a function to be complex-differentiable, it must satisfy the Cauchy-Riemann Equations

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